# ON MAXIMALLY NON-HAMILTONIAN (MNH) GRAPHS

by

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# CERTIFICATE

This is to certify that the work entitled : "ON MAXIMALLY NON-HAMILTONIAN (MNH) GRAPHS" by Manabmitra Pal has been carried out under my supervision and has not been submitted elsewhere for a degree.

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August 1982

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# ABSTRACT

In this thesis we investigated into a class of graphs, known as Maximally Non-Hamiltonian (MNH) graphs, which if recognisable in polynomial-time will imply NP = co-NP. We present this view point and indicate the existence of similar sets which also have the same property. Though we could not find any polynomial-time algorithm for the recognition of MNH graphs, our investigation resulted into enlargement of the known classes of MNH graphs, and led to a better understanding of these graphs and the problem of graph generation and recognition.

## ACKNOWLEDGEMENTS

CHAPTER 1

INTRODUCTION

Different algorithms have different time complexities.
There is wide agreement that a problem is not well-solved
until a polynomial-time algorithm is found for it. For
example, because there is no known polynomial-time algo-
rithm to recognise Hamiltonian graphs or to determine whether
a given number is composite or not, these problems till now
are considered intractable ([1]).

A large class of decision problems for which no poly-
nomial-time algorithm exist, however, can be solved in poly-
nomial time by a non-deterministic computer. They constitute
what is known as the class of NP-problems. More specififi-
cally, a decision problem is in NP, if given an instance of
that problem, a non-deterministic computer will be able to
find in polynomial time whether the answer is 'Yes'. But if
the answer is 'No' for a certain instance of that problem,
nothing is said about the time to find the solution. (For
formal definitions, the readers are referred to [1]). In
contrast, a decision problem is said to be in the class P,
if there exists a deterministic polynomial-time algorithm
to find whether the answer is 'Yes' or 'No'. Both the
problems cited above - whether a given graph is Hamiltonian

or whether a given number is composite, are in NP. An example of a problem that is in P is the problem to determine if a given graph is planar.

Cook in 1971 in an elegant paper ([2]) proved that one particular problem in NP, called the 'Satisfiability problem', has the property that every other problem in NP can be reduced to it by a polynomial-time algorithm. Thus if the 'Satisfiability problem' is solved with a polynomial-time algorithm, then such will be the case with every problem in NP. Cook in [2], and later Karp in [3], observed that many other NP-problems share the same property with the 'Satisfiability problem'. They constitute what is known as the class of NP-complete problems.

The question as to whether or not NP = P is still an open problem. Another related question that is also unsolved concerns another class of problems : the co-NP class. For each problem $\Pi$ in NP we can define another problem $\Pi^c$, the 'complementary problem'. Both $\Pi$ and $\Pi^c$ are defined over the same instance, and solution to a particular instance of $\Pi^c$ is 'Yes', if and only if the solution to the same instance in $\Pi$ is 'No'. Thus the complementary problem for the problem 'given a graph, is the graph Hamiltonian?' which is in NP, is the problem 'given a graph, is the graph non-Hamiltonian?'. On the

grounds that so many problems in co-NP do not seem to be in NP, one might well conjecture that NP $\neq$ co-NP. But the class P is closed under complementation, i.e., P = co-P. Thus NP $\neq$ co-NP implies P $\neq$ NP, but it might well be the case that P $\neq$ NP even though NP = Co-NP.

In fact, if there is a single NP-complete problem $\pi$ such that $\pi^c$ is in NP, then NP = co-NP ([1]). Though the complementary problem for any known NP-complete problem is not known to be in NP, there are problems in NP having their complementary problems also in NP. The composite number problem is one such example.

In this thesis we shall investigate into a class of graphs, which if recognisable in polynomial-time will imply that NP = co-NP. They are knwn as 'Maximally non-Hamiltonian' graphs. Before we give their definition let us first point out that corresponding to every NP-complete set, there exists a set, polynomial-time recognition of which will imply the same result, i.e., NP = co-NP. These sets are outcome of viewing NP-complete sets as sets having 'Kernels'. We discuss this view point below.

# The Concepts of Kernel and co-Kernel for a NP-complete set

Let us introduce the idea of the Kernel for an NP-complete set. Let there be an NP-complete set, A defined over the alphabet set $\Sigma$. Let there be a partial order R defined on $\Sigma^{*}$ such that (i) if $a_1 R a_2$, then $|a_1|$ and $|a_2|$ are polynomially related, i.e., there exist polynomial functions $f_1$ and $f_2$ such that $|a_1| \leq f_1(|a_2|)$, and $|a_2| \leq f_2(|a_1|)$, (ii) if $a_1 R a_2$ and $a_1 \in A$, then $a_2 \in A$, (iii) the relation R is verifiable in polynomial-time.

Then the Kernel Ka of the set A is defined as:

$$K_a = \left\{ a_1 \middle| a_1 \in A \text{ and there is no } a_2 \text{ such that } a_2 \neq a_1, a_2 \in A \text{ and } a_2 R a_1 \right\},$$ provided the set Ka is recognisable in polynomial-time.

This definition tries to capture the empirical observation that membership of many NP-complete sets are preserved by some simple operations. For example, a Hamiltonian graph remains Hamiltonian when any edge is added to it. Naturally, therefore, one is led to consider those Hamiltonian graphs which have the property that removal of any edge from them will render them non-Hamiltonian. We see immediately that each member of this set is a graph which simply is a Hamiltonian cycle. More formally, Kernel for the Hamiltonian graphs is defined as,

$$K_H = \left\{ G \middle| G \text{ is a graph and } G \text{ is just a circuit of n vertices where n is the order of } G \right\}.$$

The partial order R here is 'a subgraph of', i.e., $G_1$ R $G_2$ if $G_1$ is a subgraph of $G_2$.

We give two more examples to show the naturalness of the idea of a Kernel for a NP-complete set.

## k-clique

Let us consider the NP-complete set of graphs each of which is a k-clique. Here the relation R is the same as in the case of Hamiltonian graphs, and the Kernel is

$$K_C = \{(G,k) | \text{ G is a graph which contains just}$$
$$\text{a k-clique with no extra edges}\}$$

## k-vertex cover

A set of vertices $v_1$ is said to cover a graph G if each edge of G is incident on atleast one vertex in $v_1$. It is known that the set of all graphs each of which has a k-cover is NP-complete. Here the relation R is 'a super-graph of', i.e., $G_1$ R $G_2$ if $G_1$ is a supergraph of $G_2$ (i.e., $G_2$ is a subgraph of $G_1$). The Kernel is:

$$K_V = \{(G,k) | \text{ G is a graph which has a k vertex}$$
$$\text{cover but if any edge is added it}$$
$$\text{does not have a k-vertex cover}\}$$

If Berman-Hartmanis conjecture ([4]) is true, then it follows that every NP-complete set has a Kernel. The conjecture states that any two NP-complete sets are P-isomorphi to each other. Two sets A,B,A $\subseteq \Sigma^*$ and B $\subseteq$ $\Gamma^*$ are said

to be p-isomorphic if there exists an isomorphism p from $\Sigma^*$ to $\Gamma^*$ such that (i) p and $p^{-1}$ can be computed in polynomial time, and (ii) a $\in$ A iff p(a) $\in$ B and b $\in$ B iff $p^{-1}(b)$ $\in$ A.

That every NP-complete set has a Kernel if this conjucture is true can be shown in this way: we have already shown that there are NP-complete sets that have Kernels. Let A be a NP-complete set having the Kernel Ka defined with the partial order $R_A$. Let B be some other NP-complete set. If Bertman-Hartmanis conjecture is true then there is a 'p' as defined above. Let a relation $R_B$ be defined for B as $b_1$ $R_B$ $b_2$ if and only if $p^{-1}(b_1)$ $R_A$ $p^{-1}(b_2)$.

Thus $R_B$ is a partial order. Let us define $K_B = p(K_A)$. Since 'p' can be computed in polynomial time and $K_A$ is recognisable in polynomial time, $K_B$ is also recognisable in polynomial time. Let $b_1$ $\in$ $K_B$. Then $a_1 = p^{-1}(b_1)$ $\in$ $K_A$, i.e., $a_1$ $\in$ A, so that $b_1$ $\in$ B. Let there be a $b_2$ such that $b_2 \neq b_1$, $b_2$ $\in$ B and $b_2$ $R_B$ $b_1$. But this implies that there is a $a_2 = p^{-1}(b_2)$, such that $a_2 \neq a_1$, $a_2$ $\in$ A and $a_2$ R $a_1$. This is not possible as $a_1$ $\in$ $K_A$. Thus $K_B$ is also the kernel of B.

Hence, if Berman-Hartmanis conjecture is true, then every NP-complete set has a kernel.

On the other hand, whenever a set A has a kernel $K_a$, the set is in NP. For given an element $a \in A$, a non-deterministic computer will first guess an element $a_1 \in K_a$ such that $a_1 \, R \, a$. Since $a_1 \, R \, a$, by the definition of the Kernel, $|a_1| \leq f\,(\,|a|\,)$ where 'f' is some polynomial. And as $K_a$ is polynomial-time recognisable, to verify that $a_1$ is indeed in $K_a$ will take time polynomial in $|a_1|$, and so polynomial in $|a|$. Also, the relation R is verifiable in polynomial time. Thus a non-deterministic algorithm will take only polynomial time to recognise A.

The idea of co-Kernel is introduced to capture the dual of a Kernel. Let A be a NP-complete set over the alphabet $\Sigma$, and let A have a Kernel $K_a$ defined with the partial order R. Then the co-Kernel of A, co-$K_a$, is defined as,

$$\text{co-}K_a = \left\{ a_1 \,\middle|\, a_1 \notin A \text{ and there is no } a_2 \notin A, \right.$$
$$\left. a_1 \neq a_1, \text{ such that } a_1 \, R \, a_2 \right\}$$

Thus the co-kernel consists of members of the co-NP set that are in some sense, very nearly in the corresponding NP set. For example, for the Hamiltonian graphs, the co-Kernel co-$K_H$ corresponding to the Kernel, $K_H$, defined earlier in this chapter, is

$$\text{co-}K_H = \left\{ G \,\middle|\, G \text{ is not Hamiltonian but if any single} \right.$$
$$\left. \text{edge is added to it, it is Hamiltonian} \right\}$$

This co-Kernel forms what is known as the class of Maximally non-Hamiltonian (MNH) graphs.

Similarly, corresponding to the Kernels defined for the k-clique and k-vertex cover problems, co-kernels can be defined. For k-clique problem, $\text{co-}K_c$ is defined as

$$\text{co-}K_c = \{ G, K \} \mid G \text{ does not contain a k-clique}$$

but if any edge is added to it, it does}

And for vertex cover, the co-Kernel, $\text{co-}K_v$, is

$$\text{co-}K_v = \{ G, K \} \mid G \text{ does not contain a k-vertex cover}$$

but if any edge is deleted, it does}.

The co-kernel, therefore, is same to the co-NP set as the kernel is to the corresponding NP set. The difference lies in the fact that nothing is said about whether the co-kernel is polynomially recognisable or not. From our arguments placed earlier in this chapter, it is obvious that if a NP-complete set A has its co-kernel also poly-nomially recognisable, then the complement of A, viz., $A^c$ which is in co-NP, is also in NP. And as A is not only in NP but also a NP-complete set, this would imply NP = co-NP.

Thus the concepts of kernel and co-kernel give us a new angle from which to investigate the question whether or not NP = co-co-NP. We have tried this approach for the Hamiltonian graphs. Our effort was aimed to find a polynomial time algorithm to recognise the MNH graphs. Though we could

not find any such algorithm, our investigations did indicate the complexity of this problem.

In the following chapter, Chapter 2, we discuss the MNH graphs in some details. There we introduce two new familities of MNH graphs. We describe some methods of obtaining higher order MNH graphs from lower order ones, and applying these methods we were able to extend many known MNH graphs into hitherto unknown families or subfamilies of MNH graphs. In that chapter, we present an interesting relation between the maximal cycle length and the vertex degrees in an MNH graph. Also, for any graph G, we obtain a lower bound on the number of missing edges from the information about the degrees of its vertices, if G is to be an MNH graph.

In Chapter 3, the problems of generation and recognition are investigated. We present an essentially backtrack algorithm to generate MNH graphs that was used by us to find all MNH graphs of order $\leq 9$. Because of excessive memory requirement the algorithm could not be used to generate higher order MNH graphs. However, we made some simple modifications on that algorithm which allowed us to generate very quickly only non-isomorphic MNH graphs. For order $\leq 8$ this modified algorithm generated all non-isomorphic MNH graphs but it was shown to be not capable of doing so for n > 8. Still it gives a new approach to

prevent generation of isomorphic graphs and we indicated one necessary and sufficient condition that a graph must obey in order that it might be generated with this approach while no graph isomorphic to it is generated.

Considering its importance, it is natural to expect that finding a reasonably good algorithm for recognition of MNH graphs, will be difficult. We have, however, been able to find one heuristic for finding one Hamiltonian path from another, and using this, an algorithm is devised which was found to work surprisingly well on all known MNH graphs.

In the concluding chapter, we list some problems for further investigations. It is hoped that further research into these problems will help us to have a better under-standing about the general problem of graph generation and recognition, the MNH graphs, and of course, on the question "Is NP = Co-NP?".

# CHAPTER 2

## MAXIMALLY NON-HAMILTONIAN (MNH) GRAPHS

### 2-1. Introduction

In this chapter the maximally non-Hamiltonian (MNH) graphs will be described in some details. Some of the classes of MNH graphs that have been possible to identify are discussed. We also describe some of the properties of MNH graphs and some results concerning generation of higher order MNH graphs from lower order MNH graphs. We, however, could not find any property of MNH graphs that will uniquely characterise them, and at the same time can be checked in polynomial-time so that a deterministic algorithm could be devised to detect the MNH graphs.

Definition: A graph G is called maximally non-Hamiltonian (MNH) if

    (i) the graph G is not Hamiltonian, but

    (ii) if any one of the missing edge is added, it becomes Hamiltonian (or, equivalently, between every pair of non-adjacent vertices there is a Hamiltonian path).

Note: Thus in a MNH graph, two vertices u and v are non-adjacent if and only if there is a Hamiltonian path between u and v.

## Examples

A MNH graph of order 5 is shown in Figure 2-1. It is not Hamiltonian because of the presence of cut vertex 3. There are four missing edges, viz., (1,4), (1,5), (2,4), (2,5). The Hamiltonian paths between these pairs of non-adjacent vertices are:

| edge | H-path |
|------|--------|
| (1,4) | 1 - 2 - 3 - 5 - 4 |
| (1,5) | 1 - 2 - 3 - 4 - 5 |
| (2,4) | 2 - 1 - 3 - 5 - 4 |
| (2,5) | 2 - 1 - 3 - 4 - 5 |

Thus the graph of Figure 2-1 is MNH. Some other MNH graphs of small orders are shown in Figure 2-2.

## 2-2. Classification of MNH Graphs

In this section we describe some of the classes of MNH graphs that have been possible to identify. Some of the known MNH graphs are yet to/be put into any class.

## Family F1

One type of graphs that are MNH for all orders ([5]) is shown figuratively in Figure 2-3. Some typical examples are shown in Figure 2-4.

Another related type is where 3 complete subgraphs are mutually disjoint except for sharing 2-adjacent vertices as shown in Figure 2-5 and 2-6. They are also known to be MNH ([5]).
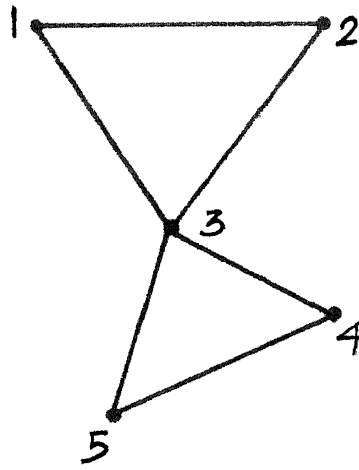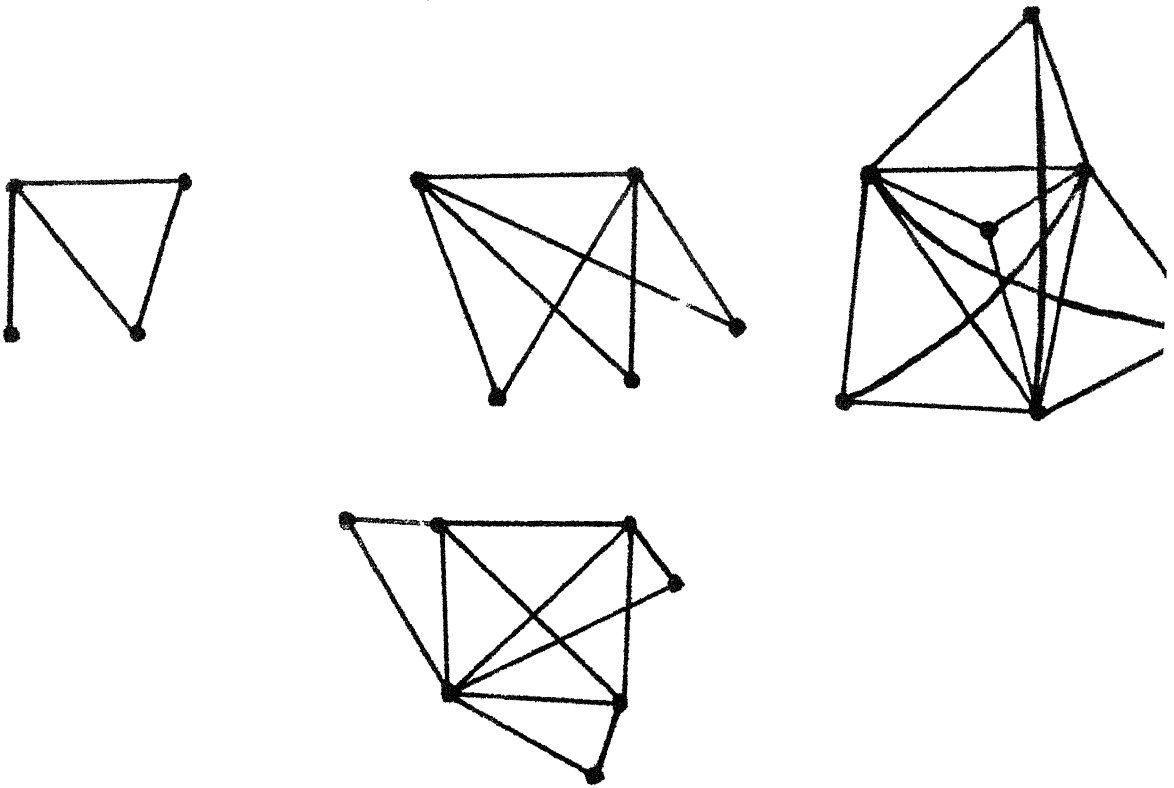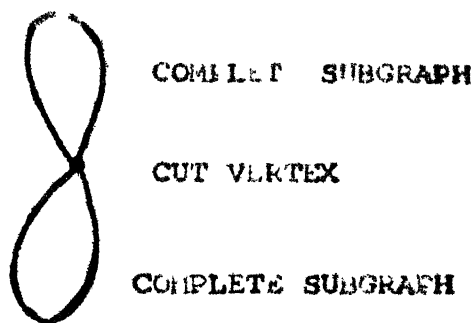
Figure 2-1.



Figure 2-2

COMPLET SUBGRAPH

CUT VERTEX

COMPLETE SUBGRAPH
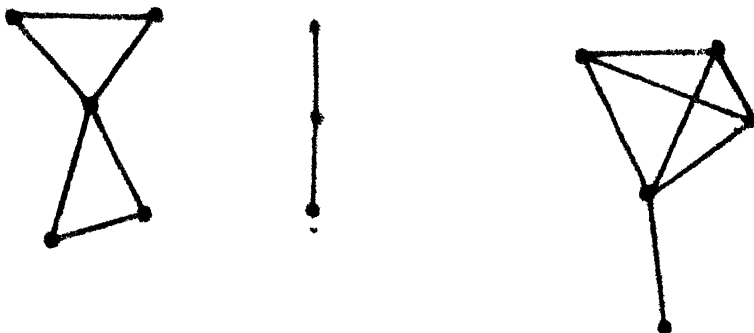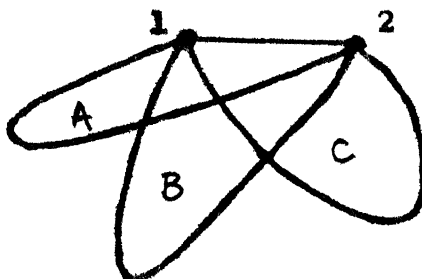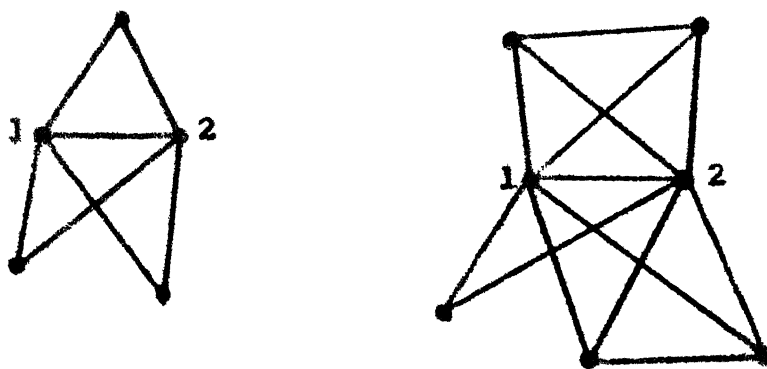
Figure 2-3

Figure 2-4

Figure 2-5

In general, whenever (p+1) complete subgraphs that
are mutually disjoint except for sharing a common complete
subgraph of p vertices constitute a graph, the graph is
MNH. Skupien has shown in [5] that these graphs consti-
tute a family of MNH graphs with the unique property of
scattering number ([6]) equalling unity.(Following Jung
in [6], the scattering number S(G) of a graph G is defin-
ed as, $S(G) = \text{maximum} \{K(G-S) - S : S \subseteq V(G) \text{ and } K(G-S) \neq 1\}$
A graph G belonging to this family has a maximum degree of (n-
n being the order of G. It contains a subgraph of order
(n-1), n>3, which is also MNH and belongs to the same family.
No graph of this family is Hypohamiltonian or homogeneously
traceable. (A graph G is called hypohamiltonian if G is not
Hamiltonian but each vertex v in G, the subgraph G-v is
Hamiltonian. Again following Skupien in [7], a graph G
is called homogeneously traceable if for each vertex v in
G, there is a Hamiltonian path beginning at v).

Family F2

A MNH graph that does not belong to Family Fl is
shown in Figure 2-7. This graph is the smallest member
of a subfamily of MNH graphs described figuratively in
Figure 2-8. There A,B,C are three complete subgraphs.
An example of this for 9 vertices is shown in Figure 2-9.
That the graph in Figure 2-9 and, in general, any member
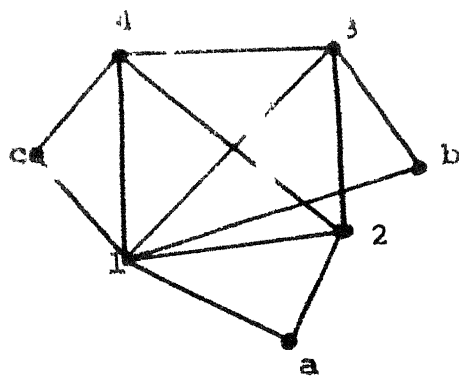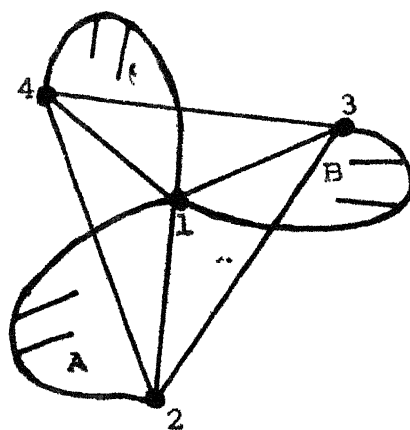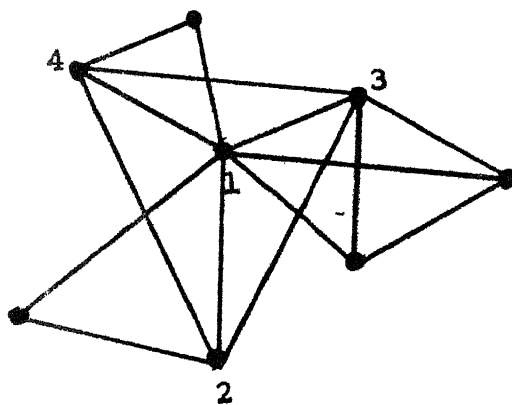
Figure 2-7



Figure 2-8



Figure 2-9

of Figure 2-8 is MNH provided the graph of Figure 2-7 is MNH, can be proved by using Theorem 2-1 to be discussed later in this chapter.

This type can be generalised into the following structure F2a. Let in a graph G there be $(2p+1)$ complete subgraphs $S_1, S_2, S_3 \ldots S_{2p+1}$. They share a complete subgraph $S_0$ of $p$ vertices. Let there be vertices $a_1, a_2, \ldots, a_{2p+1}$ on $S_1, S_2, \ldots S_{2p+1}$ respectively, forming a complete subgraph $S'$. The $p$ vertices of $S_0$ and $(2p+1)$ vertices of $S'$ forms another complete subgraph $S''$. Then G is a MNH graph. The proof follows below. Figure 2-8 corresponds to $p = 1$.

## The Proof

### G is non-Hamiltonian

If possible let there be a Hamiltonian cycle. Two cases are possible:

Case 1: In the said Hamiltonian cycle, all the vertices of each $S_i$, $1 \leq i \leq (2p+1)$, occurs consecutively.

In that path when one goes from one $S_i$ to another $S_j$, $i \neq j$, it can happen in two ways:

(a) via any one of the p vertices of $S_0$.

(b) via any edge of $S''$.

Since $S_0$ has exactly p vertices only p-times (a) can be thought of. And $S''$ is a complete subgraph of $(2p+1)$

vertices. Once any edge from S" is used, any edge adjacent to it cannot be used. But there are a maximum of p such non-adjacent edges in S". Thus (b) also can be thought of only for p-times. So the structure of the graph allows only 2p such transition between $S_i$ and $S_j$, $i \neq j$.

But if there is a Hamiltonian path where all the member of each $S_i$, $1 \leq i \leq (2p+1)$ occurs consecutively, there will be exactly (2p+1) such transitions. Hence, no such Hamiltonian path is possible.

Case 2: In the said Hamiltonian path all members of each $S_i$ does not occur consecutively.

But this implies that along the Hamiltonian path the number of transitions between $S_i$ and $S_j$, $i \neq j$, will be more than that in Case 1. So if Case 1 is an impossibility then $S_0$ is this case.

Thus the graph G is not Hamiltonian.

G is Maximally Non-Hamiltonian

Any missing edge <x,y> will be between two subgraphs $S_i$ and $S_j$, $i \neq j$ with $x \in S_i$, $y \in S_j$. Two different cases are possible. They are:

Case 1: neither x nor y is a member of S".

Case 2: one of x and y is a member of S".

In Case 1, a Hamiltonian path between x and y will require exactly 2p transitions between $S_i$ and $S_j$, i ≠ j, and that is exactly what is possible. In the second case, one among x and y, say x, is a member, say $a_i$ of S'. We note that again 2p transitions are there which allow us a Hamiltonian path between x, i.e. $a_i$ and y, where none of the transitions make use of any edge incident at $a_i$.

Thus the graph G is maximally non-Hamiltonian.

This structure F2a, will now be extended further into a more general structure that represents Family 2. In that general form, the complete subgraph S" need not contain only the vertices of $S_0$ and S'. It might contain 'extra' vertices not occurring in any of the other complete subgraphs, viz., $S_1$, $S_2$, ..., $S_{2p+1}$, $S_0$, S', so long it is a complete subgraph. That this general form represent MNH graphs will be shown later in Section 2-4, where we discuss how to generate higher order MNH graphs from lower order ones.

## Family F3

Skupien found in [6] that the graphs figuratively shown in Figure 2-10 commonly included in the class of W-M graphs are also MNH. Here A,B and C are three complete mutually disjoint subgraphs, each having atleast 3 vertices. On immediate generalisation, we find that the graphs
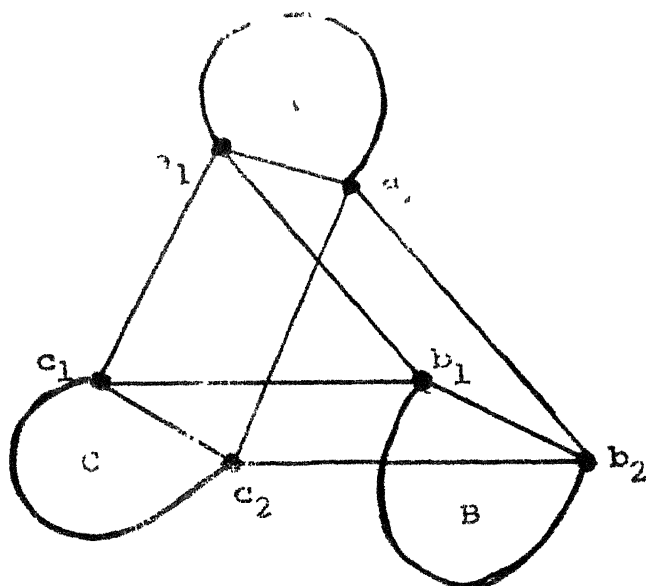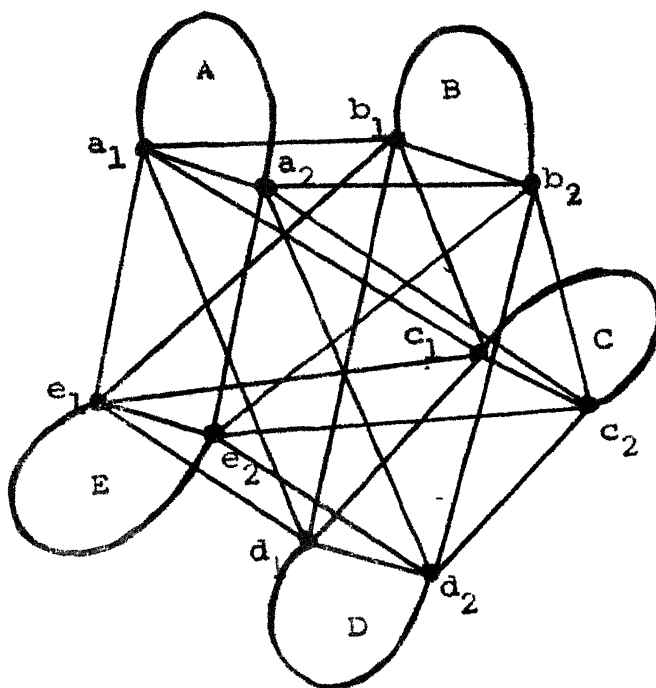
Figure 2-10



Figure 2-11

represented in Figure 2-11 are also MNH. In the Figure
2-11, A,B,C,D,E represents 5 mutually disjoint complete
subgraphs each containing atleast 3 vertices. The points
$a_1,b_1,c_1,d_1,e_1$ form a complete subgraph $S_1$ while the
points $a_2,b_2,c_2,d_2,e_2$ form another complete subgraph $S_2$.
A member $u \in S_1$ is not adjacent to a member $v \in S_2$ unless
both belong to the same subgraph - any one of A,B,C,D and E.

That it is non-Hamiltonian can be shown in this way:
Each subgraph among A,B,C,D,E contains atleast 3 vertices
and hence, atleast one 'internal' vertex, i.e., a vertex
not adjacent to any vertex from any other complete subgraph.
Also each subgraph has only two vertices which are adjacent
to any vertex outside that subgraph. Thus if there is any
Hamiltonian cycle in the graph, in that cycle all the
vertices in that subgraph must appear consecutively.
For example, to visit all the vertices in A, we have to
start at $a_1$ (or $a_2$), visit all 'internal' vertices of A,
and come out at $a_2$ (or $a_1$). So if we start at $a_1$, a
member of $S_1$, after visiting all 'internal' vertices of A,
we are at $a_2$, a member of $S_2$. Thus after the visit of any
one of the subgraphs A,B,C,D and E, we change our position
from $S_1$ to $S_2$ or from $S_2$ to $S_1$. So after visiting four
subgraphs from A,B,C,D,E, we are on the same subgraph $S_1$
where we started. If we now want to visit the remaining

subgraph, after visiting that subgraph, we will be at any of the vertices of $S_2$, from which $a_1$ cannot be reached without revisiting any vertex.

That the graph is also maximally non-Hamiltonian can be proved in a similar manner.

This idea can be extended to 7 and in general, to any odd number of complete subgraphs connected in a similar way. In that general structure say F3a, there are $(2p+1)$ mutually disjoint complete subgraphs $S_1, S_2, \ldots S_{2p+1}$, each having atleast 3 vertices. There are $(2p+1)$ vertices $a_1, a_2, \ldots, a_{2p+1}$, $a_i \in S_i$, $1 \leq i \leq (2p+1)$, forming a complete subgraph $S'$. Similarly, there are another set of $(2p+1)$ vertices, $b_1, b_2, \ldots, b_{2p+1}$, $b_i \in S_i$, $b_i \neq a_i$, $1 \leq i \leq (2p+1)$, forming another complete subgraph $S''$. Two vertices u and v, $u \in S'$, $v \in S''$ are adjacent only if they both belong to the same $S_i$ for some i. The proof is similar to the case given above, where $p = 1$.

This family F3a is further generalised into a structure that represents Family F3. In that general structure $S'$ can contain besides $a_1, a_2, \ldots, a_{2p+1}$, 'extra' vertices not occurring in any other complete subgraph, so long $S'$ remains a complete subgraph. Similarly $S''$ also can have 'extra' vertices not occuring in any other subgraph. That this general structure is indeed MNH will be shown in Section 2-4.

## Family F4

Skupien reported in [6] that the graph of order 10 shown in Figure 2-12 is a MNH graph. In Section 2-4 later in this chapter we will show how to generalise this graph into the structure shown in Figure 2-13. Here the subgraphs A and C, both have atleast 3 vertices, while the subgraphs B and D both have atleast 2 vertices.

Another type of graph very similar to that shown in Figure 2-13, is shown in Figure 2-14. This graph was found to be MNH through extensive but simple checking. Here A,C,E are three mutually disjoint complete subgraphs each having atleast 3 vertices, and B,D,F are three another mutually disjoint complete subgraphs, each having atleast 2 vertices.

Further extension of these type is incomplete but seems possible so that all of them can be included in a seperate Family F4.

## Other MNH Graphs

We will now give a description of other MNH graphs which are yet to be incorporated into some family of their own.

## A-Graphs

These were described by Skupien in [6]. He described three types - $A_1, A_2$ and $A_3$ of MNH graphs. Each one of them
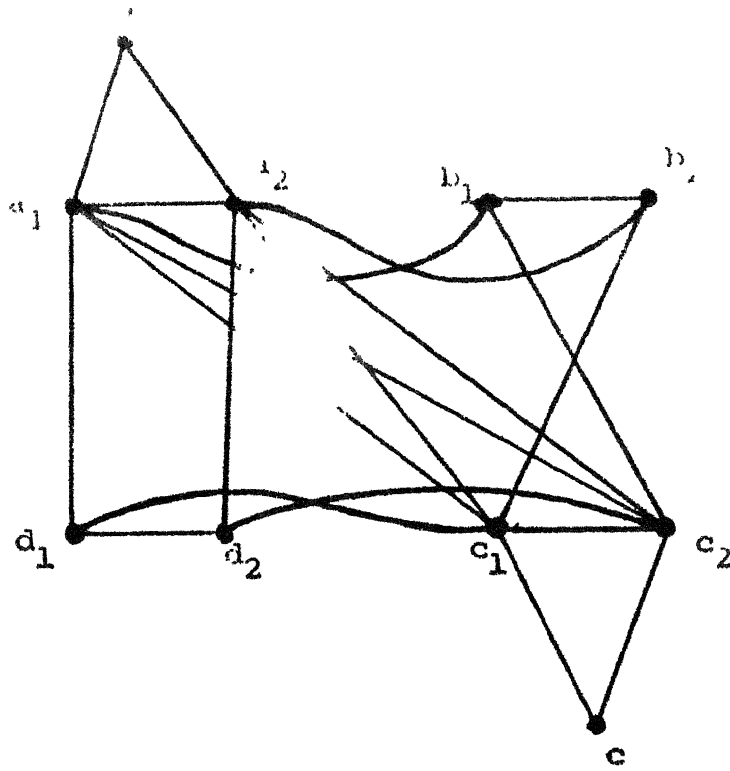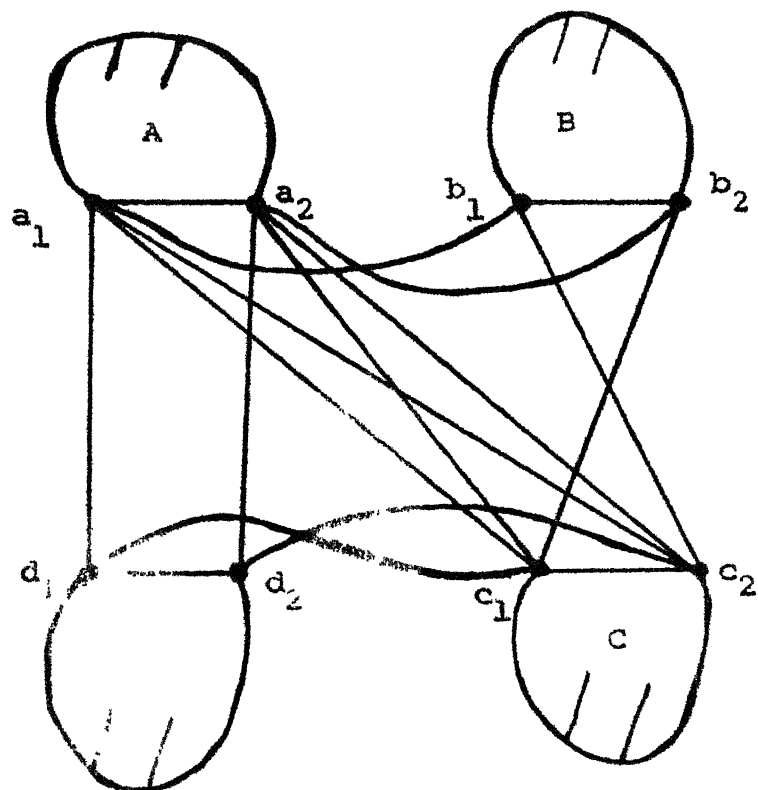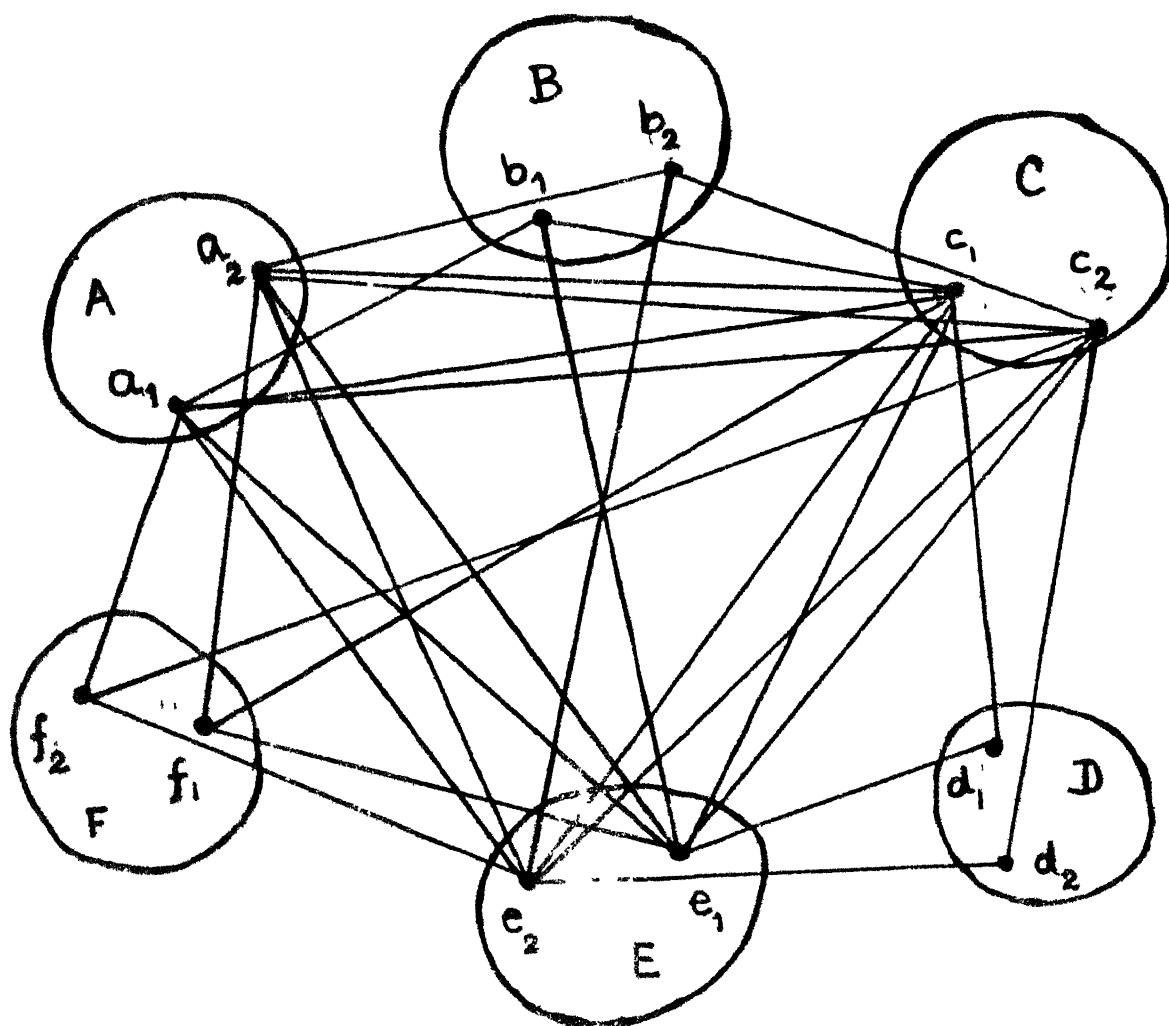
Figure 2-12



Figure 2-13

FIG 2.14

can be represented by a r x s matrix where $r = s-1$. For $A_1$, $r = 3$ and for $A_2$ and $A_3$ $r = 4$. Thus

$$A_1 = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Each such graph G contains $(r+1)$ mutually disjoint complete subgraphs, $K_{n_i}^{(i)} = 0, 1, \ldots, r$ with s vertices of $K_n^{(0)}$ labelled as $v_1, v_2, \ldots, v_s$. Moreover, each of the additional edges of G is incident to some $v_K$ according to the rule that all vertices of $K_{n_i}^{(i)}$, $i > 1$, are adjacent in G to $v_j$ if and only if $a_{ij}$ in the corresponding A is 1.

It is easy to see that $A_1$ belongs to the Family F2 discussed earlier and all the subfamilies of F2 can be represented similarly by some matrix A.

## Petersen Graph

The well known Petersen graph shown in Figure 2-16 is also a MNH graph, as pointed out in [6]. In Section 2-4 later we extended this graph into the structure shown figuratively in Figure 2-16(a)

## Another Graph

Figure 2-15 shows another graph of order 10 which does not belong to any one of the subfamilies discussed so far ([6]). This graph, however, was extended into a subfamily by using Theorem 2-3 to be discussed later in this chapter.

## 2-3. General Properties of MNH Graphs

The following three properties of an MNH graph G were proved in [5] and [6].

Property 1: $1 \leq \chi(G) \leq (n-1)/2$, where $\chi(G)$ is the connectivity and n is the order of the graph G.

Property 2: If u and v are two vertices and $<u,v>$ denotes the edge connecting them, $E(G)$ the set of edges in G, then

$$<u,v> \notin E(G) \Rightarrow \text{degree } [u] + \text{degree } [v] < n$$

Property 3: $s(G) \leq 1$ where $s(G)$ denotes the scattering number of the graph G.

Note: Scattering number ([8]) $s(G)$ of a graph G is defined as follows:

$$s(G) = \text{maximum} \left\{ K(G-S) - S : S \subseteq V(G) \text{ and } K(G-S) \neq 1 \right\}.$$
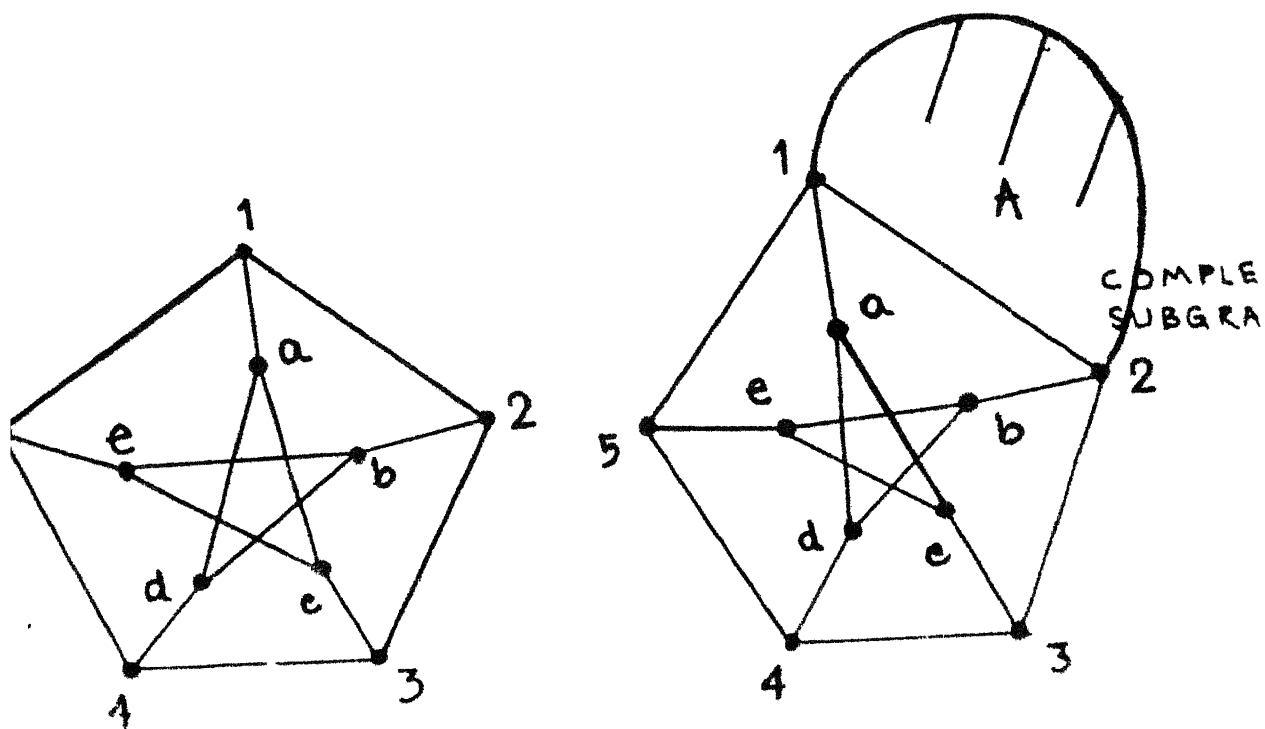
FIG 2-15



COMPLE
SUBGRA
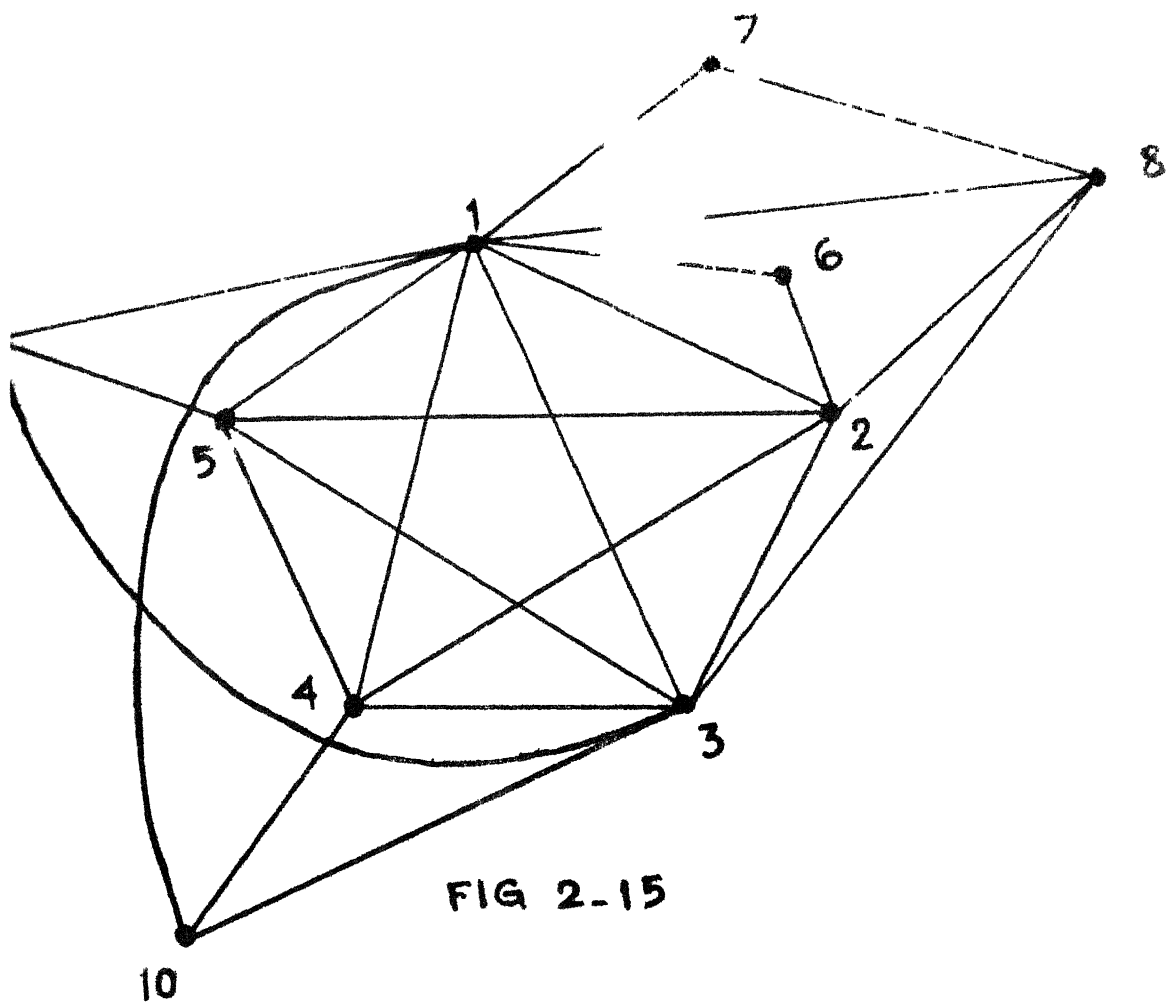
FIG 2-16(a)

Some other properties that we found are discussed below.

Property 4: It is not true that for all values of n, each MNH graph of order n contains a subgraph G' of order (n-1) such that G' is also a MNH graph.

Note: Two exceptions are (i) the Petersen graph and (ii) the graph of Figure 2-7.

Property 5:

(a) If in a MNH graph G of order n there is no cycle of length $\geq$ (n-L), L $\geq$ 1, there is no vertex of degree (n-L-1) in G.

(b) If in a MNH graph G of order n, the maximal cycle length is (n-L) where L $\geq$ 2, then there are no vertices of degrees (n-L), (n-L+1), ... (n-2) in G.

Proof:

(a) If possible let x be a vertex in G having degree = (n-L-1). Since L $\geq$ 1, there is atleast one vertex u to which x is not adjacent. As G is a MNH graph, this implies that there is a Hamiltonian path in G between x and u.

Let in that path, of all the vertices to which x is adjacent, closest along that path, to u be y. That is, while going from x to u, from y onwards all vertices are non-adjacent to x in G. But minimum length from x

to y is (n-L-1), and as x is adjacent to y, the minimum
length of this cycle is (n-L). Thus there is a cycle of
length $\geq$ (n-L), which is contradictory to our assumption.

Thus there is no vertex of degree (n-L-1) in G.

(b)   If the maximal cycle length is (n-L) where $L \geq 2$,
then there is no cycle of length (n-L+i) for $i \geq 1$. Now
with $i \leq L - 1$, i.e., $1 \leq (L-i)$, i.e., with $1 \leq i \leq (L-1)$,
using (a), there is no vertex of degree (n-L+i-1). Thus
there is no vertex of degree (n-L),(n-L+1), ..., (n-2).

Note:   The converse of these results are not true. For
example, in Peterson graph it is easy to check
that while all the vertices have degree = 3, the
maximal cycle length = 9 = n - 1. Thus though
there is a cycle of length (n-1), there is no
vertex of degree n-2. Again, though there is no
vertex of degree 4,5, ..., n the maximal cycle
length is not 4.

## 2-4.  How to obtain Higher Order MNH Graphs from known Lower Order ones?

In this section we will discuss some results concerning
generation of one MNH graph from another.

### Theorem 2-1.

If in a MNH graph G, there is a vertex v such that
it is connected to a set of vertices S(v) such that any
vertex belonging to S(v) is adjacent to all other vertices

of $S(v)$, then if we replace $v$ by two vertices $v_1$ and $v_2$ such that

(i) $v_1$ and $v_2$ are adjacent,

(ii) the set $S(v_1)$ of vertices to which $v_1$ is adjacent $= S(v) = S(v_2) =$ the set of vertices to which $v_2$ is adjacent, then the resulting graph $G'$ is also a MNH graph.

## Proof

### $G'$ is non-Hamiltonian

If possible let there be a Hamiltonian cycle in $G'$. If in this path $v_1$ and $v_2$ are adjacent, then obviously the same cycle with $v_1, v_2$ replaced by $v$ existed in $G$ too. But $G$ is a MNH graph, and so in the said cycle in $G'$, $v_1$ and $v_2$ cannot occur consecutively.

If $v_1$ and $v_2$ are not adjacent in the given cycle, then using conditions (i) and (ii) we can get another Hamiltonian cycle where $v_1$ and $v_2$ appears consecutively, as shown in Figure 2-17. 'And this is already shown to be not possible. Hence, $G'$ is not Hamiltonian.

### $G'$ is Maximally non-Hamiltonian

All the missing edges $\langle x, y \rangle$ in $G'$ can be classified into two types:

type 1: neither $x$ nor $y$ is $v_1$ or $v_2$.

type 2: either $x$ or $y$ is $v_1$ or $v_2$.
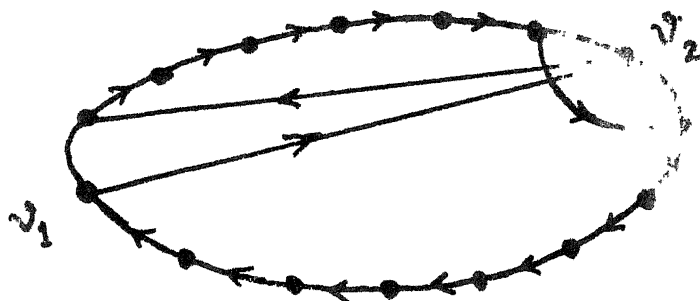
We will deal the two types seperately.

FIG 2-17

## type 1

Since neither x nor y is $v_1$ or $v_2$, they occurred in G too, and were non-adjacent there also. But G being a MNH, there is a Hamiltonian path between x and y in G. Obviously a similar path exists in G' too between x and y where everything remaining the same, instead of $v$, $v_1$ and $v_2$ appears consecutively.

## type 2

Let $x = v_1$. Thus $v_1$ is not adjacent to y in G'. So by condition 2, in G v is not adjacent to y. But G being MNH, there is a Hamiltonian path between v and y in G. Using conditions (i) and (ii) a similar path must exist in G', with $v_2$ appearing after $v_1$.

Thus between every pair of non-adjacent vertices there is a Hamiltonian path. Hence, the graph G' is MNH.

The converse of the Theorem 2-1 is also true.

## Theorem 2-2

If in an MNH graph G, $v_1$ and $v_2$ are two vertices such that

1. $v_1$ and $v_2$ are adjacent,
2. the set $S(v_1)$ = vertices adjacent to $v_1$

$$= S(v_2)$$

$$= \text{vertices adjacent to } v_2$$

3. $S(v_1) = S(v_2)$, forms a complete subgraph, i.e., every member in $S(v_1)$ is connected to every other member

member of $S(v_1)$, then the graph G' obtained by merging $v_1$ and $v_2$ into a single point v is also MNH.

Proof

Similar to that of Theorem 2-1.

Applications of Theorem 2-1

We present some examples here where by applying Theorem 2-1 we can get MNH graphs of larger order from a MNH graph of smaller order.

Example 1

It is known that the graph G in Figure 2-18 is MNH. Here the vertex v satisfies the conditions (i) and (ii) of Theorem 2-1. Thus replacing v by two vertices $v_1$ and $v_2$ we get the graph G' in Figure 2-19 which is also MNH. And by repeated application of the same theorem we arrive at the general struccure of Figure 2-5.

Example 2

In the graph of Figure 2-7 each one of the vertices a,b,c satisfies conditions (i) and (ii) of Theorem 2-1. So by repeated application of this Theorem we can arrive at the general structure of Figure 2-8.

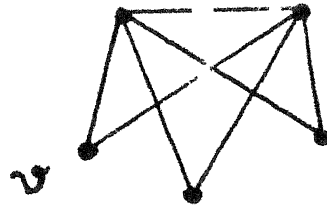A more general way of obtaining an MNH graph of order n from another MNH graph of order (n-1) will now be discussed

FIG 2-18
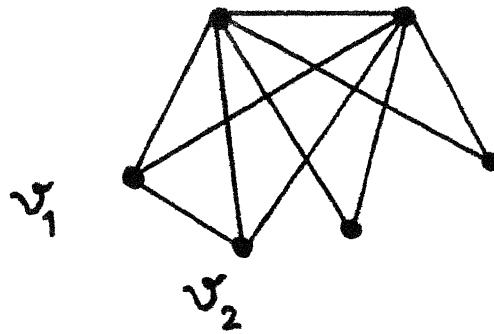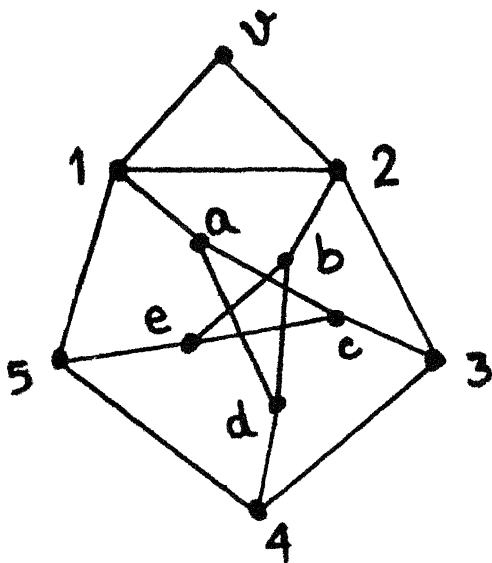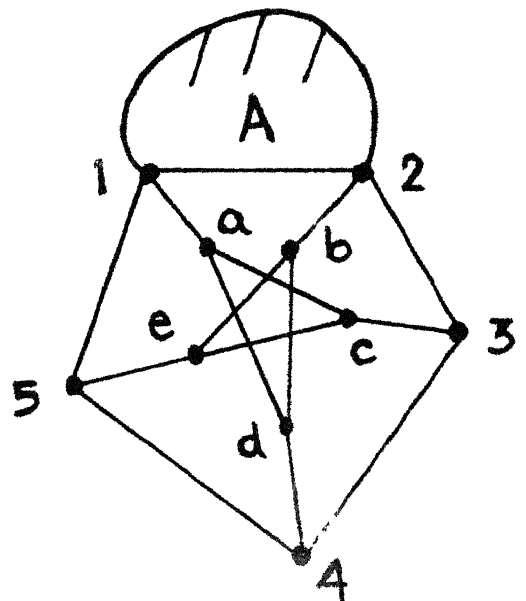


FIG 2-19



FIG 2-20



FIG 2-21

Let G be an MNH graph of order n. In it let $G_s$ be
one maximal complete subgraph in the sense that every
two members of $G_s$ are mutually adjacent, and no vertex
outside $G_s$ can be included in $G_s$ without losing this
property. Also, suppose that for every pair of non-
adjacent vertices there be atleast one Hamiltonian path
in which two vertices from $G_s$ occur successively (equiva-
lently, the Hamiltonian path uses atleast one edge from
$G_s$). If we augment G to G' by adding one vertex v in such
a way that v in G' is adjacent to all the vertices in $G_s$
but to none outside $G_s$, then G' is also MNH.

The correctness of the above observation can be shown
very simply, by first showing that G' is non-Hamiltonian
and then G' is maximally non-Hamiltonian. We use this idea
to make some useful generalisations on some known MNH graphs.

Example 1: Augmenting Petersen Graph

In case of the Peterson graph (Figure 2-16) it can be
verified by extensive checking that any two adjacent
vertices forms such a $G_s$. Taking $G_s = \{1, 2\}$, we can get
another graph G', as shown in Figure 2-20, which is also
an MNH graph. We can use now Theorem 2-1 repeatedly to
get a more general structure (Figure 2-21) which is MNH.
ThereAis a complete subgraph of order $\geq 2$.

However, in Figure 2-21 now no two adjacent vertices
from the original Peterson graph are found to satisfy the

requirement to form a $G_s$. This can be shown in this way.
Let us consider (2,3) as a possible candidate for $G_s$.
Consider a Hamiltonian path between 2 and a. If in that
path 2 and 3 are successively visited, the internal
vertices of A cannot be visited. Similar is the case
for the sets $\{1,a\}$ and $\{1,5\}$. Consider now any edge which
is not adjacent to <1,2>, for example <3,4>, and consider
$\{3,4\}$ as a possible candidate for $G_s$. There is a Hamil-
tonian path between 1 and 4. We have to leave 5 if we
want to visit all the internal vertices of A, and vice
versa. All other edges are in the same position as
<3,4>. So the structure represented in Figure 2-21 cannot
be further generalised using the arguments presented alone.

Example 2 : Augmenting Graphs in Family F2

    While describing the family F2, the subfamily F2a
was proved to be MNH. In F2a, it can be shown that the
subgraph S" satisfies the two conditions stipulated for
$G_s$. Thus it was possible to generalise it into the larger
subfamily F2.

Example 3 : Augmenting Graphs in Family F3

    In the subfamily F3a, both S' and S" can be shown
to satisfy the requirements for $G_s$. Hence, this subfamily
was extended into the bigger family F3.

Example 4 : Augmenting the graph of Figure 2-12

This graph has been found to be MNH by Skupien in $[6]$. Here the sets $\{a_1, a_2, a\}$ and $\{c_1, c_2, c\}$ are possible condidates for $G_s$, as each of them has an 'internal' vertex to visit which an edge form it must be used in all the Hamiltonian paths. Thus we can use the arguments presented earlier to augment this graph by adding vertex to either of these sets. Again by extensive checking it is possible to show that both the sets $\{b_1, b_2\}$ and $\{d_1, d_2\}$ also satisfy the requirements for $G_s$. So we can add vertex b to the graph such that it is adjacent to $b_1, b_2$ only, and vertex d which is adjacent to $d_1$ and $d_2$ only. This process can be repeated to get the general structure shown in Figure 2-13. In that general structure both the subgraphs A and C have atleast 3 vertices while both the subgraphs B and D have atleast 2 vertices.

The requirement that atleast one edge from $G_s$ should occur in atleast one Hamiltonian path between every pair of non-adjacent vertices, is very difficult to verify in general. In the following theorem, some conditions on $G_s$ are imposed which though sufficient are not necessary. This theorem can be viwed as a more generalised version of Theorem 2-1.

Theorem 2-3

Let in an MNH graph G there is a set $G_s$ of vertices such that

(i) $G_s \in V(G)$, the vertex set of graph G,

(ii) All members in $G_s$ are mutually adjacent,

(iii) For each $x \notin G_s$, x is not connected to all members in $G_s$ (in this sense the set $G_s$ is maximal on condition (ii)),

(iv) (a) there is atleast one vertex in $G_s$ which is not connected to any vertex outside $G_s$, or (b) there are atleast two vertices in $G_s$ each of which is connected to only one vertex outside $G_s$, or both (a) and (b).

Then if we add a vertex v to G such that in the augmented graph G', v is adjacent only to vertices belonging to $G_s$ but not to any vertex outside $G_s$, the resulting graph G' is also an MNH graph.

Proof

We will show that $G_s$ satisfies the requirement that every Hamiltonian path in G uses atleast one edge from $G_s$. This, in turn, implies that not only G' is non-Hamiltonian it is also MNH.

We will discuss the two conditions seperately.

## Condition (iv)(a)

Let a be a vertex in $G_s$ such that it is not connected to any vertex outside $G_s$. In all the Hamiltonian paths in G, where a does not occur as any one of the end vertices, to visit a, two edges incident/a on, that are wholly in $G_s$, are used. Again in all the Hamiltonian paths where a occurs as an end vertex, one edge incident on a, and hence, wholly in $G_s$ is going to be used. So in either case all the Hamiltonian paths use one edge from $G_s$.

## Condition (iv)(b)

Let a and b the two vertices in $G_s$ satisfying (iv)(b). Since a and b are adjacent there cannot be any Hamiltonian path between them as G is MNH. So in every Hamiltonian path atleast one among a and b does not occur as the end vertex. And to visit that vertex among a and b which does not occur as an end vertex, the Hamiltonian path must use an edge from $G_s$ because of condition (iv)(b).

Hence, the proof.

## Application of Theorem 2-3

We present here an example of obtaining an MNH graph of order n from another one of order n-1, which could not be done by using Theorem 2-1.

## Example 1

It was noted earlier in Section 2-2 that the graph of Figure 2-15 is an MNH graph. Here $G_s = \{1,2,3,4,5\}$ with a = 5 and b = 4. On augmentation by a vertex v we get the graph shown in Figure 2-22. The same theorem can be applied repeatedly to obtain larger graphs.

### 2-5. On the Number of Missing Edges in an MNH Graph

In this section we present a lower bound on the number of missing edges in an MNH graph.

Let G be an MNH graph of order n. We consider two cases:

Case 1: In G, is a ~~in every~~ pair of non-adjacent vertices (u,v), atleast one of the vertices has degree 1.

Case 2: In G, in all pairs of non-adjacent vertices (u,v), both u and v have degree > 1.

If Case 1 occurs, it can be easily shown that in that case G must contain a complete subgraph G' of (n-1) vertices and a separate vertex u of degree 1 which is connected to some vertex of G'. Thus, the number of missing edges exactly equals (n-2).

In Case 2, let (u,v) be a pair of non-adjacent vertices. So there is a Hamiltonian path between u and v. Let it be $u\, u_1 u_2 \ldots u_{n-2} v$. Since degree [u] > 1, there is an edge a form u incident on some $u_i, 2 < i \leq (n-2)$.

FIG 2-22

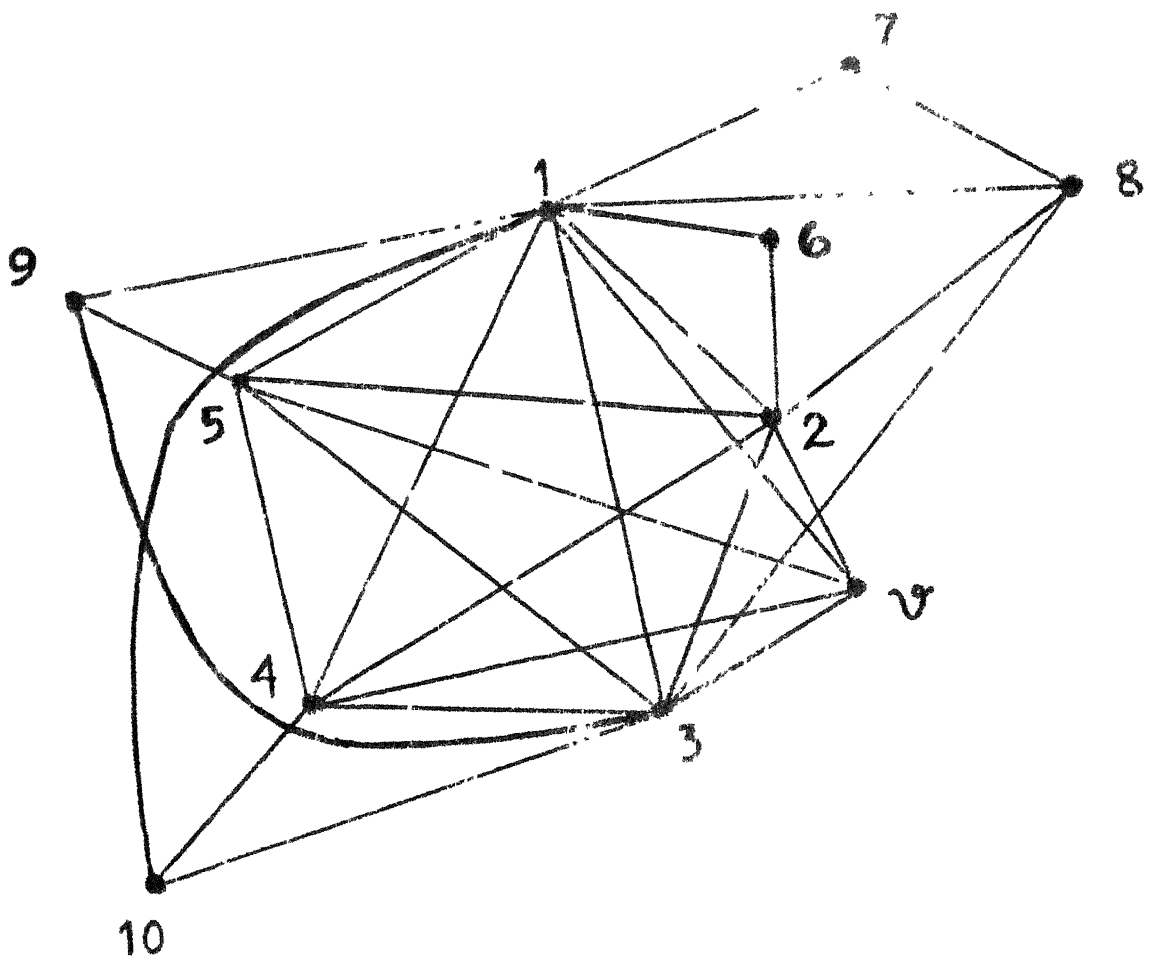Similarly there is an edge b from v to some $u_j$, $2 \leq j < (n-2$

Here two cases might happen:

Case (a) : $i < j$, i.e., the edges a and b do not 'cross' each other.

Case (b) : $i > j$, i.e., the edges a and b do cross each other.

If case (a) happens then there is a Hamiltonian path between $u_{i-1}$ and $u_{j+1}$, the path being $u_{i-1}$ $u_{i-2}$ $\cdots$ $u_1$ $u_i$ $u_{i+1}$ $\cdots$ $u_j$ v $u_n$ $u_{n-2}$ $\cdots$ $u_{j+1}$. Thus G being MNH, the edge $<u_{i-1}, u_{j+1}>$ cannot be there in G. Thus the existence of such a pair of edges implies the non-existence of an edge in G. And for obvious reasons, if there are two such pairs of edge each satisfying case (a), each of them will imply the non-existence of a different edge in G. Or in other words, if there are m such pairs, m edges must be missing from G.

In case (b) also, through similar argument it can be shown that if there are m such pairs, that will imply the non-existence of m edges in G.

Now the total number of such pairs, either 'crossing' or 'not crossing' is (degree [u] - 1)(degree [v] - 1). Of this half must be of case (a) or case (b) only. So in G, the number of missing edges > (degree [u] - 1) (degree [v] - 1)/2. And maximally over all such pairs of

vertices, we conclude that the number of missing edges

in G > maximum $\left\{ (\text{degree } [u]-1)(\text{degree}[v]-1)/2 \right\}$ u,v are

nonadjacent , (which is trivially true

for case 1 also)

# CHAPTER 3

## GENERATION AND RECOGNITION OF MNH GRAPHS

An essentially backtrack algorithm was deviced to gene-
rate all MNH graphs of order upto 8. Though the runtime was
appreciably small, it takes memory of the order of n! to
generate MNH graphs of order n, and hence, could not be
used for larger number of vertices. We also present an
algorithm to recognise MNH graphs based on a method to
generate one Hamiltonian path from another Hamiltonian path.

## Generation Algorithm

The essential idea of this algorithm is to go on adding
edges till no further edge can be added without making the
graph Hamiltonian.

## Addition of edges

The edges are numbered according to a simple rule.
For n = 5, this is shown with the following matrix A:

|   | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 1 |   | 5 | 6 | 7 |
| 2 | 5 |   | 8 | 9 |
| 3 | 6 | 8 |   | 10 |
| 4 | 7 | 9 | 10 |   |

Figure 3-1.

where

> $a_{ij}$ = edge connecting the ith vertex with the
> jth vertex.

In general for any n, it can be shown that

$$a_{ij} = j - n + i(2n - i - 1)/2 \quad \text{where } i < j$$

To ensure that the algorithm will cover all possible
graphs the edges are to be added and deleted according to
an order. Let us consider a tree where the nodes are diffe-
rent edges; the root is edge '1'; the leafnodes correspond
to edge n(n-1)/2; and each node corresponding to edge
$1 \leq K \leq n(n-1)/2$ has sons corresponding to edges (k+1),
(k+2), ..., n(n-1)/2. With every node 'k' we can also
associate a graph having edge k and all its ancestors. It
is easy to see that each node will give a different graph
and the total number of nodes = $2^{n(n-1)/2}$ =total number of
graphs possible with n vertices.

The algorithm traverses this tree in a simple DFS
manner. When at a particular node it finds that none of
its sons can be added to it without making the graph Hamil-
tonian, it knows the graph being generated is non-Hamiltonian.
It then checks whether the graph is also MNH, and if it
finds so it prints the graph.

## Ensuring Non-Hamiltonianness

To check non-Hamiltonianness an extensive list of all possible cycles, viz., $(n-1)!/2$ cycles were generated and kept as an array of vertices. To each cycle a count was associated which gies the number of edges in that cycle presently in the graph. Each edge occurs in $(n-2)!$ cycles and whenever an edge is added, the counts for the corresponding cycles are incremented by one. Whenever the count for a cycle is $(n-1)$, the algorithm finds out that edge 'k', which if added, would have completed the cycle, and marks it as 'critical'. Instead of marking that edge as just critical through a boolean variable, it marks critical $[k] = L$, where $L$ is the number of edges added to the graph so far. This helps to find out when deleting a certain edge the edge which were 'critical', but now would be harmless because of deletion of the present edge.

## Maximality

We could not find any way to make the algorithm generate only maximally non-Hamiltonian graphs. To know whether a graph generated is maximal, the algorithm checks, before printing, all the absent edges. If their 'critical' values are all greater than zero, it prints the graph.

## Modifications

Two modifications helped to reduce the runtime. They are discussed below.

(a) Some of the graphs are going to be isomorphic to each other. Some method was necessary to reduce the number of graphs generated by not generating graphs isomorphic to some graph already generated.

We added the condition that at every stage of generation the vertex degree sequence must be descending. Thus when trying to add a certain edge 'k' that connects vertices u and v, it ensured that even after addition of k, degree [u] $\leq$ degree [u-1] and degree [v] $\leq$ degree [v-1].

Though this condition helped to generate all non-isomorphic MNHs of order upto 8, this method is not foolproof. There exist classes of isomorphic graphs which cannot be generated with this restriction. We cite two examples below, one of them is the Petersen graph which is known to be an MNH graph.

## Example 1 (Figure 3-3)

The dgree sequence is 6 5 4 3 3 3 2 and it is descending. Before the addition of the edge 8, which connects vertices 2 and 4, the degree sequence is 6 1 1 1 1 1 1 which is descending. But after the addition of edge 8, the degree sequence is 6 2 1 2 1 1 1, which is not descending. So
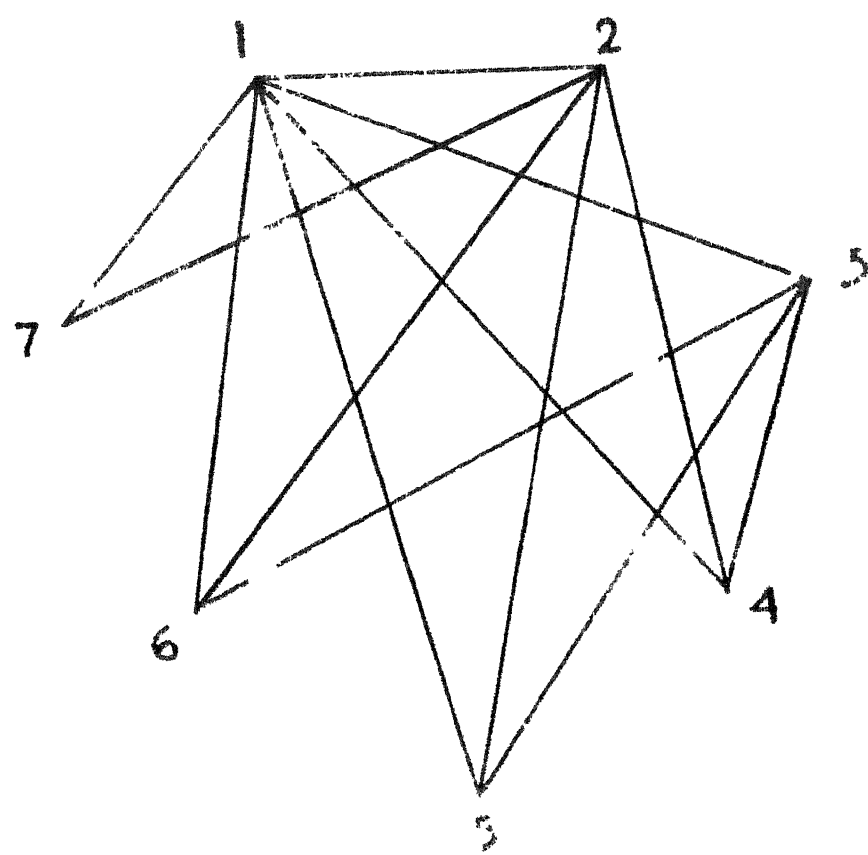
FIG 3 5

atleast, this graph is not generated with the said restriction.

Let us see whether there exists a graph G which is isomorphic to the graph above but generatable with the said restriction. G must have descending degree sequence and, in fact, the same degree sequence 6 5 4 8 3 3 2. G, therefore, will have the same vertex labelling as the graph above for vertices 1, 2, 3 and 7. The vertices 4,5,6 for G will be just a permutation of those of the graph above. But it is easy to check that irrespective of how the vertices 4,5,6 are labelled, the degree sequence stops remaining descending when edge 8 is added.

Example 2   The Petersen Graph (Figure 2-16)

The Petersen graph has no clique of order 3. Also all vertices has degree 3. We tried to generate graphs of order 10, with these two conditions, but Petersen graph was found not generatable.

However, we can cite a condition that are both necessary and sufficient, that a graph G has to obey if it going to be generated with the said restriction.

Let G be a graph of order n with vertices labelled $v_1, v_2, v_3, \ldots, v_n$. Thus G is generatible by the generating algorithm with the said restriction if and only if

1. The degree sequence is descending, i.e.,

   degree $[v_1] \geq$ degree $[v_2] \geq \ldots \geq$ degree $[v_n]$

   2. For each pair of vertices $v_i$ and $v_j$, $i < j$, and
for $k$, $1 \leq k \leq (i-1)$, the number of vertices from $v_1, v_2,$
$\ldots, v_k$ connected to $v_i$ is greater or equal to the number
of vertices from $v_1, v_2, \ldots, v_k$ connected to $v_j$.

## Proof

### Necessary Condition

The first condition follows obviously. The second
can be proved in this manner : Let m = number of the edge
between $v_k$ and $v_j$. Before any edge numbered > m were added,
the degree sequence was still descending. But at this stage
degree $[v_i]$ = number of vertices from $v_1, v_2, \ldots, v_k$ to which
$v_i$ is adjacent in the completed graph G, and degree $[v_j]$ =
number of vertices from $v_1, v_2, \ldots, v_k$ to which $v_j$ is adjacent
in G. Hence condition 2 follows.

### Sufficient condition

Let us assume that the graph G satisfying conditions 1
and 2 is not generatable. So there exists an edge m such
that (1) if all edges $\geq$ m were deleted from G, the degree
sequence is still descending, but the addition of edge m
violates the degree sequence, (2) m is the smallest possible
number satisfying (1).

Let G' be the graph obtained from G by deleting all edges $\geq$ m and let (G'+e) denote the graph with the edge m added to G'. Let the edge m connect vertices $v_i$ and $v_j$, i < j. Addition of edge m can violate the degree sequence in two ways (in the following lines degree* ( i ) denotes the degree of the ith vertex in (G'+e)).

    1. degree*$(v_{i-1})$ < degree*$(v_i)$, i > 1.

But degree* $(v_{i-1})$ = degree $(v_{i-1})$ in G and

    degree* $(v_i)$ $\leq$ degree $(v_i)$ in G.

So this implies that

    degree $(v_{i-1})$ < degree $(v_i)$ in G, which contradicts the assumption (condition 1) that the degree sequence is descending.

    2. degree* $(v_{j-1})$ < degree*$(v_j)$

As the edge m is between vertices $v_i$ to $v_j$, this cannot happen if i = j - 1, as before the addition of edge m the degree sequence was still descending. So i < j - 1.

Now, degree* $(v_{j-1})$ = number of vertices from $v_1$, ... $v_i$ connected to $v_{j-1}$ in G, and similarly degree*$(v_j)$ = number of vertices from $v_1$ ... $v_i$ connected to $v_j$ in G. That it implies that

    No. of vertices from $v_1$ ... $v_i$ connected to $v_{j-1}$ <

    No. of vertices from $v_1$ ... $v_i$ connected to $v_j$.

Thus this violates the condition 2.

(b) Another modification was added to prevent the algorithm generate any disconnected graph. If when trying to add an edge K that connects $v_i$ and $v_j$, $i < j$, it is found that degree $[v_i] = 0$, and for all $K > i$, degree $[v_k] = 0$ the graph going to be generated is disconnected. As modification (a) is also there, all that was necessary was to check whether degree $[v_i] = 0$.

## Recognition Algorithm

If a graph G is MNH, then two vertices u and v are non-adjacent if and only if there is a Hamiltonian path between u and v. The algorithm presented here is based on a method which finds Hamiltonian path between one pair of vertices from a Hamiltonian path between another pair.

## The Basis

Let in a graph G, there be a Hamiltonian path between $u_1$ and $u_n$, the path being $u_1 u_2 \ldots u_n$. Then if $u_1$ is adjacent in G to some $u_j$, $j > 2$, then there is a Hamiltonian path between $u_{j-1}$ and $u_n$, the path being $u_{j-1} u_{j-2} \ldots u_1 u_j u_{j+1} \ldots u_n$. Similarly, if $u_n$ in G is adjacent to $u_k$ for some $k < n - 1$, then there is a Hamiltonian path between $u_1$ and $u_{k+1}$, the path being $u_1 u_2 \ldots u_k u_n u_{n-1} \ldots u_{k+1}$.

Thus from a given Hamiltonian path between $u_1$ and $u_n$ we might find Hamiltonian paths between $u_1$ and some other vertices, and between $u_n$ and some other vertices. The

process can be repeated on all the new Hamiltonian paths found till no new pair of vertices between which Hamiltonian path exists can be found. Our recognition algorithm uses this technique to find some Hamiltonian paths without much labour.

## The Algorithm

With this method an algorithm to recognise graphs could be like this -

Step 1: Check whether the given graph is Hamiltonian or not. If 'Yes' then stop, else proceed to the next step.

Step 2: Try to find a pair of non-adjacent vertices (u,v) such that no Hamiltonian path between them is yet found. If no such pair found, stop, as the graph is MNH, else proceed to next step.

Step 3: Try to find a Hamiltonian path between u and v. If no Hamiltonian path could be found between u and v, stop (as the graph is non-Hamiltonian but not MNH).

Step 4: Use the method described to find Hamiltonian paths between new pairs of vertices till no new such pair is found.

Step 5: Go to Step 2.

For all classes of MNH known to us, it was found that
we can in this way generate one Hamiltonian path between
every pair of non-adjacent vertices starting from a given
arbitrary Hamiltonian path. If this is true for all MNH
graphs (a fact that we conjecture but cannot prove), then
the recognition algorithm could be simplified considerably
as shown below:

Step 1: Try to find a pair of non-adjacent vertices u and v.
        If no such pair is found, stop (the graph is complete,
        hence Hamiltonian).

Step 2: Try to find a Hamiltonian path between u and v. If
        no such path found, stop (the graph can be either
        Hamiltonian or non-Hamiltonian but certainly not MNH).

Step 3: Using the method, find Hamiltonian paths between
        new pairs of vertices, till no new such pair is found.

Step 4: If any of the pair found in Step 3 has adjacent
        vertices, then stop (as the graph is certainly
        Hamiltonian).

Step 5: If Hamiltonian path between every pair of non-
        adjacent vertices is found then the graph is MNH.

Step 6: Stop.

That this algorithm will indeed recognise MNH graphs can be shown in this way: If an input graph G is MNH, the algorithm will not stop at Step 1, Step 2 or Step 4, for obvious reasons. And as we assumed the correctness of our conjecture that the method finds in any MNH graph every pair of vertices between which a Hamiltonian path exists, in Step 5, G will be recognised as MNH. Conversely, let G be a certain graph the algorithm 'recognises' as MNH in Step 5. This means that in G between every pair of non-adjacent vertices there is a Hamiltonian path. This can happen only when G is either MNH or a supergraph of some MNH graph G'. But if our conjecture is true, then in any supergraph of an MNH graph also the method will be able to find every pair of vertices between which a Hamiltonian path exists. And if G is a supergraph of some MNH graph atleast one of this pair must have adjacent vertices, which will be detected in Step 4. So G must be MNH.

Note on the Method

This method to find a Hamiltonian path between one pair of vertices from a Hamiltonian path between another pair, however, cannot succeed in all graphs to find every such pairs of vertices. We cite two examples, none of which is MNH.

Example 1 (Figure 3-4)

In the graph of Figure 3-4, starting from the Hamil-
tonian path between 1 and 14, viz., 1 2 3 4 5 6 7 8 9 10
11 12 13 14, it is possible, using this method, to find
Hamiltonian paths between vertices 1 and 9, 6 and 14,
and 6 and 9, but not between vertices 3 and 12, 3 and 10,
6 3 and 12, and 3 and 10.

Example 2 (Figure 3-5)

In this graph too, starting from the Hamiltonian path
between 1 and 19, viz., 1 2 3 4 5 6 7 8 9 10, 11  12  13
14  15  16  17  18  19  it is not possible to find any
Hamiltonian path between 13 and 19, though such a path
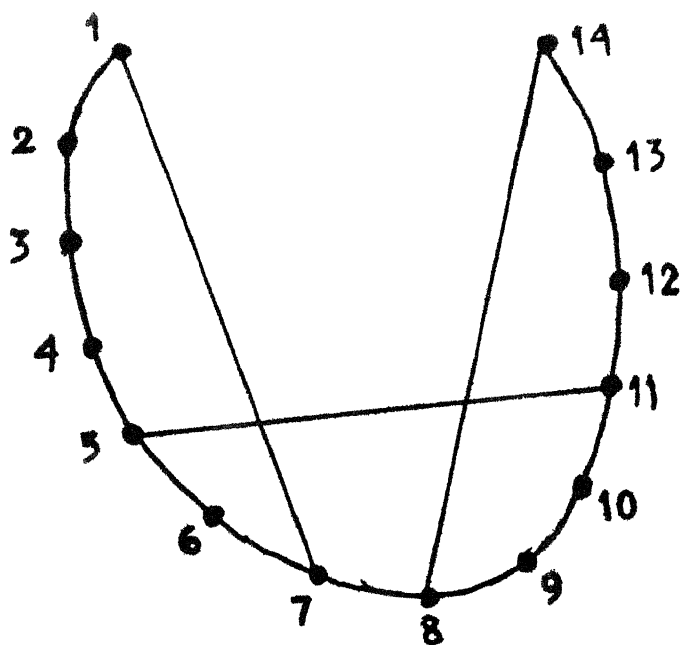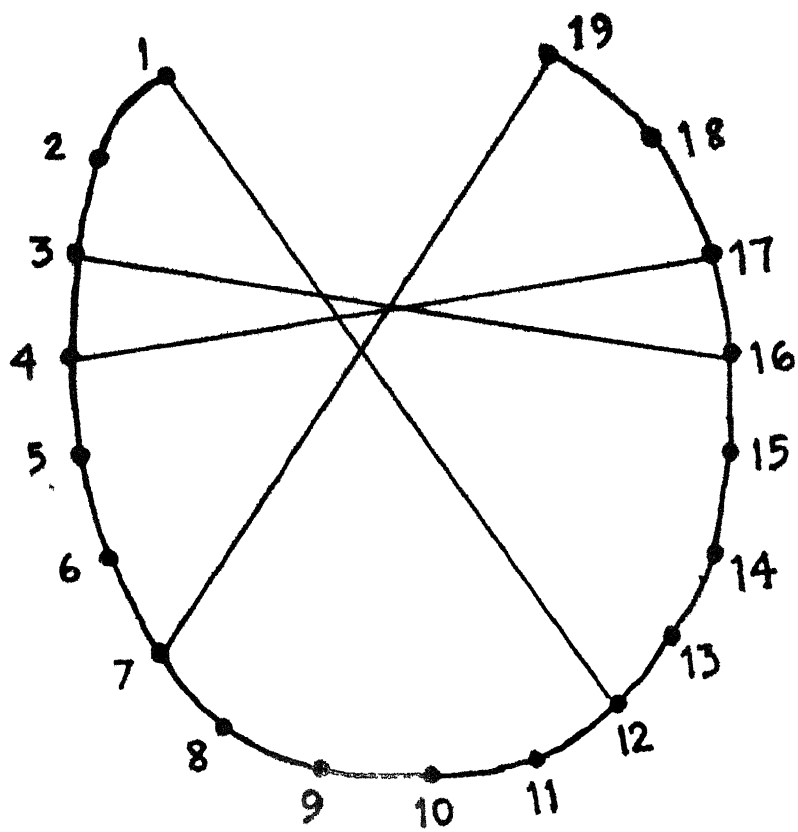exists, viz., 13, 14  15  16  3  2  1, 12 11 10 9 8 7 6
5 4 17 18 19.

FIG. 3-4



FIG 3-5

# CHAPTER 4

## CONCLUSION

In this thesis we have introduced the concepts of kernel and co-kernel for the NP-complete sets. As we explained in Chapter 1 this presents a new approach to the problem whether or not NP = co-NP. It was shown that if Berman-Hartmanis conjecture on p-isomorphism is true then every NP-complete set has such a kernel and co-kernel, and the polynomial-time recognisability of any of these co-kernel will imply NP = co-NP. We have identified kernels and co-kernels for some NP-complete sets, and shown that the set of MNH graphs forms a co-kernel for the Hamiltonian graphs.

In the Chapters 2 and 3 we investigated the MNH graphs with a view to finding a polynomial-time algorithm to recognise them. Though we could not find any such algorithm our investigations have given us a better understanding of these graphs. In Chapter 2 it has been possible to enlarge the known classes of MNH graphs. We also presented there some results concerning the number of missing edges, the maximal cycle length and vertex degrees in an MNH graph. In the Chapter 3 we presented our algorithm to generate and recognise MNH graphs. Though the recognition algorithm

algorithm was not polynomial time complexity,
it was indeed a satisfaction to find that it works quite
fast for all the known MNH graphs.

Before concluding, we will list some of the other
problems that can be pursued in future:

1. To see whether our conjecture in Chapter 3, i.e.,
the method presented there for finding a Hamiltonian path
between one pair of vertices from a Hamiltonian path between
another pair, can find all such pairs in an MNH graph if
started from some arbitrary Hamiltonian path, is true.
In particular, to find necessary and sufficient conditions
for a graph G such that given one Hamiltonian path in it,
a Hamiltonian path between every pair of vertices between
which atleast one Hamiltonian path exists, can be found out.

2. To find one Hamiltonian path in a given MNH graph.
We suspect that this problem is simpler than the general
problem of finding a Hamiltonian path in an arbitrary graph,
if it exists.

3. To devise a better algorithm for generating MNH
graphs, i.e., an algorithm which requires less memory but
possibly more time. It seems possible that the method
used for the recognition algorithm can be used for generating
one MNH graph from another MNH graph.

4. To investigate the co-kernel for other NP-complete
problems, and their relation.

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| 12 | 4 | 10 | 11 |
| 12 | 9 | 10 | 11 |
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